Quantum Chaos and Random Matrix Theory - Some New Results

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Abstract

New insight into the correspondence between Quantum Chaos and Random Matrix Theory is gained by developing a semiclassical theory for the autocorrelation function of spectral determinants. We study in particular the unitary operators which are the quantum versions of area preserving maps. The relevant Random Matrix ensembles are the Circular ensembles. The resulting semiclassical expressions depend on the symmetry of the system with respect to time reversal, and on a classical parameter $\mu = \mathrm{tr}U - 1$ where U is the classical 1-step evolution operator. For system without time reversal symmetry, we are able to reproduce the exact Random Matrix predictions in the limit $\mu \to 0$. For systems with time reversal symmetry we can reproduce only some of the features of Random Matrix Theory. For both classes we obtain the leading corrections in μ. The semiclassical theory for integrable systems is also developed, resulting in expressions which reproduce the theory for the Poissonian ensemble to leading order in the semiclassical limit.

I. INTRODUCTION

One of the most remarkable achievements of quantum chaology, was the observation that spectra of systems which are chaotic in the classical limit, obey universal statistics which follow the predictions of Random Matrix Theory (RMT) [1]. In most cases, the above correspondence was established by comparing a few statistical functions, mostly two point densities, or functions derived from them like the Δ_3 or the Σ^2 statistics. The nearest neighbor spacing distribution $P(s)$ is another example. The study of n-points ($n > 2$) spectral statistics which would test the quantum chaos - RMT correspondence more closely, are much more difficult to implement in practice.

In a previous paper [2] we proposed a different measure, which is based on the study of the statistical properties of the *spectral determinant*, sometimes referred to as the *secular function*, the *spectral ζ function* or the *characteristic polynomial*: It is the function which vanishes if and only if its argument belongs to the energy spectrum. For matrices of dimension N the characteristic polynomial is

$$p_H(x) = \det(Ix - H) = \prod_{n=1}^{N}(x - \lambda_n(H))) = \sum_{l=0}^{N} a_l(H)x^l. \tag{1}$$

The statistical properties of this function, can be expressed in terms of the statistics of either the eigenvalues λ_n or the coefficients a_l. Since the two sets of variables are functionally related, they are statistically equivalent. In practice, however, one cannot check the full spectral distribution, and therefore it is advantageous to study statistical measures which are based on other accessible quantities. The measure which was the subject of [2] was the autocorrelation function

$$C(\xi) = \mathcal{N}\left\langle \int_{-\infty}^{+\infty} w(x)p_H(x+\xi/2)p_H(x-\xi/2)\mathrm{d}x \right\rangle_H \tag{2}$$

where $w(x)$ is a positive function with a finite support and unit integral. $\mathcal{N}$ is a normalization constant so that $C(0) = 1$. Analytical expressions for $C(\xi)$ were derived in [6] and [2] for the standard RMT ensembles. These results can also be written down in terms of $\langle |a_l|^2 \rangle_H$.

There are other advantage to the study of the statistics of the spectral determinant, which are especially important for establishing the connection between quantum chaos and RMT. The Gutzwiller trace formula, which gives the semiclassical theory for the spectral density, is not a proper function. The derivation of spectral statistics based on this theory, should therefore be augmented by additional assumptions [3] or truncation procedures [5]. In contrast, the semiclassical expression for the spectral determinant involves a finite number of periodic orbits, and therefore it converges on the real energy axis. The semiclassical spectral determinant preserves another important property, namely, it is explicitly real for real energies ([7], [8], [9]). Thus, the semiclassical study of the statistical properties of the spectral determinant can be based on a relatively solid starting point. Last but not least, the semiclassical spectral determinant shares many of its properties with the Riemann Siegel expression for the Riemann ζ function on the critical line. The autocorrelation function for the Riemann ζ is the subject of a recent study of Cheung and Keating [10]. The rigorous results obtained for the Riemann ζ case, provide some support to the physically reasonable, (yet mathematically uncontrolled) approximations made in the semiclassical theory to be discussed here.

The behavior of $C(\xi)$ can be intuitively clarified, by considering two extreme cases: An equally spaced (infinitely rigid) spectrum produces a correlation function which is a strictly periodic function of ξ,

$$C(\xi) = \cos \pi\xi \tag{3}$$

Where ξ is measured in units of the mean level spacing. On the other hand, a Poissonian spectrum with N spectral points, yields a positive correlation which decays to zero on a scale which is proportional to $\sqrt{N}$. Thus, the lack of correlation between the energy levels, induces a slowly decaying correlation function. The canonical random ensembles all display level repulsion which induce strong correlations. It is expected, therefore, that the "incipient crystalline character" [11] of the spectrum of these canonical random matrix ensembles will manifest itself by oscillatory, yet slowly decaying correlation functions. The more rigid the

spectrum, the more marked and persistent will be the oscillations of $C(\xi)$.

In the present work we would like to concentrate on the spectra of Unitary operators, and use the autocorrelation function of their spectral determinant to study the correspondence between quantum chaos and RMT. The spectral statistics of Unitary operators appear naturally in quantum chaology, when one studies, e.g., Floquet operators corresponding to time periodic hamiltonians, [12], [13] or scattering matrices [14]. The quantization of classical area preserving mappings also involve quantum unitary operators. [15] In cases where the classical phase space is compact, the corresponding quantum Hilbert space is finite, and then one deals with unitary matrices of finite dimension. This is the case we shall study here. We shall provide a new derivation of the results obtained in [2], and add some new results.

In the next chapter we shall define the correlation function for the cases of interest here, and quote the RMT results which were obtained recently by the Essen group [6]. We shall then turn to the semiclassical theory and assuming the system to be classically integrable or chaotic, we shall establish the conditions under which correspondence with Poissonian or RMT can be achieved.

II. SOME RESULTS FROM THE THEORY OF RANDOM MATRICES

The spectrum of a $N\times N$ unitary matrix S consists of N unimodular eigenvalues $e^{i\theta_l}$, $1 \leq l \leq N$. It is convenient to write the characteristic polynomial so that it is real on the unit circle

$$Z_S(\omega) = e^{\frac{i}{2}(N\omega-\Theta)}\det(I - e^{-i\omega}S) \tag{4}$$

Where $e^{i\Theta} = \det(-S)$. The characteristic polynomial can be written down as

$$Z_S(\omega) = e^{\frac{i}{2}(N\omega-\Theta)}\sum_{l=0}^{N} a_l e^{-i\omega l} \tag{5}$$

The unitarity of S leads to the relations

$$e^{-i\Theta/2}a_l = e^{i\Theta/2}a^*_{\Lambda-l} \tag{6}$$

The autocorrelation function reads now

$$C_\beta(\xi) = \mathcal{N}_\beta \frac{1}{2\pi}\int_0^{2\pi} \langle Z_S(\omega+\xi/2)Z_S(\omega-\xi/2)\rangle_\beta \, \mathrm{d}\omega = \frac{\sum_{l=0}^{N} < |a_l|^2 >_\beta e^{i\xi(l-\frac{N}{2})}}{\sum_{n=0}^{N} < |a_n|^2 >_\beta} \tag{7}$$

where $< \cdot >_\beta$ stands for the average with respect to the spectral measure of the circular ensembles of interest: the orthogonal (COE, $\beta = 1$), the unitary (CUE $\beta = 2$) or the Poissonian ($\beta = 0$) ensembles. The $< |a_l|^2 >_\beta$ are the coefficients in the Fourier expansion of the correlation functions. In the sequel we shall develop the semiclassical theory for these coefficients.

In the semiclassical derivation, we shall make use of the following identities:

$$\det(I - xS) = \sum_{n=0}^{N} a_n x^n = \exp\left(-\sum_{k=1}^{\infty} \frac{x^k}{k} s_k\right) \tag{8}$$

where $s_n = \mathrm{tr} S^n$. We define the generating function

$$G_\beta(x,y) = \left\langle \det(I - xS)\det(I - yS^\dagger)\right\rangle_\beta = \left\langle \exp\left(-\sum_{k=1}^{\infty}\frac{1}{k}(x^k s_k + y^k s_k^*)\right)\right\rangle_\beta \tag{9}$$

so that

$$< a_n a_m^* >_\beta = \frac{1}{n!m!}\frac{\partial^{n+m}}{\partial^n x \partial^m y} G_\beta(x,y)\bigg|_{x,y=0} \tag{10}$$

The most important message is that the building blocks of the generating function are the traces $s_n = \mathrm{tr} S^n$.

The ensemble averages $< |a_n|^2 >_\beta$ were calculated in [6] for all values of β:

$$< |a_n|^2 >_{\beta=1} = 1 + \frac{n(N-n)}{N+1} \quad ; \quad < |a_n|^2 >_{\beta=2} = 1. \tag{11}$$

For the Poissonian ensemble one gets [6]

$$< |a_n|^2 >_{\beta=0} = \begin{pmatrix} N \\ n \end{pmatrix} \tag{12}$$

It is convenient to introduce the scaled correlation length $\eta = \xi \frac{2\pi}{N}$ and in the limit of large N the sums in (7) can be approximated as integrals, to give

$$C_{\beta=1}(\eta) = \frac{3}{2}\left(\frac{\sin\pi\eta}{\pi\eta} + \frac{1}{\pi^2}\frac{\partial^2}{\partial\eta^2}\frac{\sin\pi\eta}{\pi\eta}\right) \quad ; \quad C_{\beta=2}(\eta) = \frac{\sin\pi\eta}{\pi\eta} \tag{13}$$

The correlation function for the Poisson ensemble reads

$$C_{\beta=0}(\eta) = \cos^N\frac{2\pi}{N}\eta \approx \left(1 - \frac{1}{2}(\frac{2\pi}{N}\eta)^2\right)^N \approx \exp\left(-\frac{2\pi^2\eta^2}{N}\right). \tag{14}$$

The Poissonian correlation function decays much slower than the correlation functions of the other ensembles.

III. THE SEMICLASSICAL THEORY

The quantum unitary operator $\hat{S}$ which we consider, is assumed to be the quantum analogue of an area preserving map $\mathcal{M}$ acting on a finite phase space domain of area A. (For the present discussion we shall confine our attention to maps with the twist property. The semiclassical treatment can be extended to the general case). The phase space coordinates are denoted by $\gamma = (q, p)$. The map can be defined in terms of a generating function which is the action $\Phi(q, q')$

$$p = -\frac{\partial\Phi(q,q')}{\partial q} \quad ; \quad p' = \frac{\partial\Phi(q,q')}{\partial q'}. \tag{15}$$

The explicit mapping function $\gamma' = \mathcal{M}(\gamma)$ is obtained by solving the implicit relations (15).

The semiclassical expression for the matrix elements of $\hat{S}$ in the q representation is

$$< q|S|q' >= \left(\frac{1}{2\pi i}\right)^{\frac{1}{2}}\left[\frac{\partial^2\Phi(q,q')}{\partial q\partial q'}\right]^{\frac{1}{2}} \mathrm{e}^{i\Phi(q,q')/\hbar} \tag{16}$$

In the semiclassical limit, Λ, the dimension of the Hilbert space where $\hat{S}$ acts, is the integer part of $\frac{A}{2\pi\hbar}$.

We have seen above, that one can express the coefficients a_l in terms of the traces $s_n = \mathrm{tr}S^n$. The semiclassical approximation for s_n involves the periodic manifolds of the classical map. For hyperbolic maps [16],

$$s_n = \mathrm{tr}S^n \approx \sum_{t\in P_n} \frac{g_t n_t e^{ir(\Phi_t/\hbar - \nu_t\frac{\pi}{2})}}{|\det(I - M_t^r)|^{\frac{1}{2}}} \tag{17}$$

P_n is the set of all primitive periodic orbits of M, with periods n_t which are divisors of n, so that $n = n_t r$. Orbits which are related by a discrete symmetry are counted once, and their multiplicity is denoted by g_t. M_t is the monodromy matrix and ν_t is the Maslov index. The action along the periodic orbit is

$$\Phi_t = \sum_{j=1}^{n_t} \Phi(q_j, q_{j+1}) \quad (\text{with} \;\; q_{n_t+1} = q_1). \tag{18}$$

For integrable maps, we use the phase space variables (I, ϕ) where I is the classical invariant. The domain of the mapping is $I \in [I_{min}, I_{max}], \phi \in [0, 2\pi]$. In this representation, the action takes the form $\Phi(\phi, \phi') = \Phi(\phi' - \phi)$. The explicit form of the map is

$$I' = I \quad ; \quad \Delta\phi = \phi' - \phi = F(I) \tag{19}$$

Where $F(I)$ is the solution of the generating relation $I = \frac{\mathrm{d}\Phi(\Delta\phi)}{\mathrm{d}\Delta\phi}$. The expression for $\mathrm{tr}S^n$ is now

$$\mathrm{tr}S^n = s_n \approx \sum_{m=1}^{n} \left[\frac{2\pi}{n\hbar F'(I_{n,m})} \right]^{1/2} \mathrm{e}^{i\left[n\Phi(\Delta\phi = 2\pi\frac{m}{n})/\hbar - (n+\frac{1}{2})\frac{\pi}{2}\right]} \tag{20}$$

Where the summation is carried over the periodic manifolds of period n and winding number m. They occur at values of I which satisfy $F(I_{n,m}) = 2\pi\frac{m}{n}$.

Before we can proceed any further, we must clarify an essential point. In contrast with the RMT, the semiclassical theory deals with a *single* system, which does not depend on any random parameter which could represent an ensemble. However, averaging is mandatory in order to get a meaningful theory since the quantities we calculate fluctuate appreciably. We generate the "semiclassical ensemble" of s_n, (with $1 \leq n \leq \Lambda/2$) by considering the inverse Planck constant as a parameter, and different realizations of the ensemble are distinguished by the value of the parameter $\hbar^{-1}$. We restrict $\hbar^{-1}$ to the interval $|\hbar^{-1} - \hbar_0^{-1}| < \Delta$. The mean value $\hbar_0^{-1}$ is large enough to justify the use of the semiclassical approximation. That is, for typical orbits $\hbar_0^{-1}|\Phi_t - \Phi_{t'}| >> 1$. The interval Δ is taken to be small on the scale of $\hbar_0^{-1}$, but sufficiently large so that $\Delta|\Phi_t - \Phi_{t'}| > 2\pi$. In this way, the phases (mod2π) of the semiclassical expressions can be considered random. The averaging over the "semiclassical ensemble" is effected by

$$< A >_\hbar = \frac{1}{\Delta}\int_{\hbar_0^{-1}-\Delta/2}^{\hbar_0^{-1}+\Delta/2} \mathrm{d}\hbar^{-1} A(\hbar^{-1}). \tag{21}$$

With this definition of the ensemble average, we get for both classically integrable and chaotic maps,

$$< s_n >_\hbar = 0. \tag{22}$$

The variance for the classically chaotic case reads,

$$< |s_n|^2 >_\hbar \approx \sum_{t\in P_n} \frac{g_t^2 n_t^2}{|\det(I - M_t^r)|} \tag{23}$$

For integrable maps we get

$$< |s_n|^2 >_\hbar \approx \frac{2\pi(I_{max} - I_{min})}{2\pi\hbar} = \Lambda \tag{24}$$

Thus, the $\hbar^{-1}$ averaging provides the well know diagonal (random phase) approximation. Due to the exponential proliferation of periodic orbits in hyperbolic maps, action difference become smaller as n increase, which is the reason why the diagonal approximation is not valid uniformly. We shall make use of the diagonal approximation in the restricted range $n < \Lambda/2 \approx \hbar_0^{-1}$ where the diagonal approximation is justified. For integrable maps, the variance of s_n is independent of n. The result $\frac{1}{\Lambda} < |s_n|^2 >_\hbar \approx 1$ implies that the spectral two-point correlation function for integrable systems is Poisson [3].

At this point we would like to make a crucial observation:

(♠) *The semiclassical ensemble of $\{s_n\}$ is an ensemble of independent random Gaussian variables.*

We shall show that this is approximately valid as long as the values of n are kept bellow the dimension Λ which is of order $\hbar_0^{-1}$. Consider the correlator $< (s_n)^k (s_m^*)^l >_\hbar$. If n and m are relatively prime, the actions Φ_t which contribute to s_n and s_m are sufficiently different. Thus, all the terms in the product $(s_n)^k (s_m^*)^l$ are oscillatory and will yield a vanishing result upon averaging. If n and m have a common divisor, j, choose $k = m/j, l = n/j$, and all the amplitudes which involve repetitions of the primitive orbits of length j will

contribute non oscillatory terms to the correlator. However, for hyperbolic maps, the number of periodic orbits which involve repetitions is exponentially smaller than the total number of periodic orbits, and the statistical independence of the variables s_n and s_m is ensured. The corresponding approximation is more difficult to justify for integrable maps because the proliferation of periodic manifolds is only algebraic.

If we check all other correlators using the approximation that repetitions can be neglected, we find that the s_n are Gaussian random variables. It should be noted that RMT implies that in the limit of large dimensions, (which coincides with the semiclassical limit) the traces are random Gaussian variables [6]. This property, which is shared by the statistical and the semiclassical ensembles, has far reaching consequences, and it constitutes a strong link between RMT and quantum chaos.

Having established the statistical properties of the semiclassical ensemble we can calculate the generating function (9) for this ensemble.

$$\begin{aligned} G_\hbar(x,y) &= \left\langle \det(I-xS)\det(I-yS^\dagger)\right\rangle_\hbar \\ &= \left\langle \exp\left(-\sum_{k=1}^{\infty}\frac{1}{k}(x^k s_k + y^k s_k^*)\right)\right\rangle_\hbar \\ &= \exp\left(\sum_{k=1}^{\infty}(xy)^k \frac{<|s_k|^2>_\hbar}{k^2}\right) \end{aligned} \tag{25}$$

so that

$$< a_n a_m^* >_\hbar = \delta_{n,m}\,\frac{1}{n!}\frac{\partial^n}{\partial^n v} Z_\hbar(v)\bigg|_{v=0} \tag{26}$$

with

$$Z_\hbar(v) = \exp\left(\sum_{k=1}^{\infty} <|s_k|^2>_\hbar \frac{v^k}{k^2}\right) \tag{27}$$

The coefficients in the Taylor expansion $Z_\hbar(v) = 1+\sum_{k=1}^{\infty} A_l v^l$ can be obtained by taking successive derivatives of the two expressions for $Z_\hbar(v)$

$$A_l = \frac{1}{l}\sum_{k=1}^{l} A_{l-k}\frac{<|s_k|^2>_\hbar}{k} \tag{28}$$

We would like to emphasize the following important points

• One can use (28) to calculate $< |a_l|^2 >_\hbar = A_l$ in the range $1 \leq l \leq \Lambda/2$ *exclusively*. We have to impose this restriction because the derivation which leads to (28) is not exact. The approximations destroy the delicate balance between the $< |s_k|^2 >_\hbar$ which is necessary to maintain the symmetry $< |a_l|^2 >_\hbar =< |a_{\Lambda-l}|^2 >_\hbar$ which follows from unitarity. Because of the same reason, $A_l \neq 0$ for $l > \Lambda$, which contradicts the basic ingredient of the theory, namely, that we deal with a characteristic polynomial of order Λ.

• Because of the reasons listed above,

$$C(\xi) \neq \mathrm{e}^{-i\xi/2\Lambda} Z_\hbar(\mathrm{e}^{i\xi}) \tag{29}$$

• The input necessary to compute $< |a_l|^2 >_\hbar$ consists of all the $< |s_k|^2 >_\hbar$ with $k \leq l \leq \Lambda/2$. Thus, the semiclassical ensemble of s_n can be confined to the range $n \leq \Lambda/2$ where the arguments which support the conjecture (♠) can be most trusted.

The semiclassical expressions for $< |s_k|^2 >_\hbar$, depend on the underlying classical dynamics (23),(24). We shall treat the various cases separately.

A. Classical chaotic dynamics

It is instructive to consider the Fredholm determinant for the *classical* evolution (Frobenius Peron) operator

$$U(\gamma, \gamma') = \delta(\gamma' - \mathcal{M}(\gamma)), \tag{30}$$

where γ is a phase space point in the domain of $\mathcal{M}$. A straight forward integration gives

$$u_n = \mathrm{tr} U^n = \sum_{t \in P_n} \frac{n_t g_t}{|\det(I - M_t^r)|}. \tag{31}$$

Comparing this expression with (23), we may write

$$< |s_n|^2 >_\hbar \approx \sum_{t \in P_n} \frac{g_t^2 n_t^2}{|\det(I - M_t^r)|} =< g_t n_t >_{cl(n)} u_n \quad , \tag{32}$$

where for any phase space function $\rho(\gamma)$, $< \rho >_{cl(n)}$ is the approximate phase-space average of ρ obtained by sampling ρ on the set of periodic points of period n with the appropriate

weights. As $n \to \infty$, $< \rho >_{cl(n)}$ approaches the true phase space average of ρ, since $u_n \to 1$ in this limit [17]. Defining

$$g = \lim \frac{< g_t n_t >_{cl(n)}}{n} \tag{33}$$

we get

$$< |s_n|^2 >_\hbar \approx g n u_n \tag{34}$$

Thus, (27) can be written as

$$\begin{aligned} Z_\hbar(v) &= \exp\left(\sum_{k=1}^{\infty} < |s_k|^2 >_\hbar \frac{v^k}{k^2}\right) \\ &\approx \exp\left(\sum_{k=1}^{\infty} g u_k \frac{v^k}{k}\right) \\ &= \exp\left(-g \mathrm{tr} \log(\mathcal{I} - vU)\right) \\ &= \left(\det(\mathcal{I} - vU)\right)^{-g} \\ &= \left(\zeta_{Ruelle}(v)\right)^g \end{aligned} \tag{35}$$

Where $\zeta_{Ruelle}(z)$ is the Ruelle ζ function for the classical mapping $\mathcal{M}$. It is defined in terms of the Fredholm determinant of the classical evolution (Frobenius Peron) operator by

$$\zeta_{Ruelle}(z) = \left(\det(\mathcal{I} - zU)\right)^{-1}. \tag{36}$$

The relation (35) is one of the central results of the present work, and we shall come back to it after obtaining explicit expressions for $< |a_m|^2 >_\hbar$.

Starting with the second line in (35) we can easily derive the recursion relations

$$< |a_m|^2 >_\hbar = \frac{g}{m} \sum_{k=1}^{m} < |a_{m-k}|^2 >_\hbar u_k \tag{37}$$

which should be solved with the initial condition $< |a_0|^2 >_\hbar = 1$.

Let consider systems which are strongly mixing and for which all transients die out on a short time scale. In such cases, we may replace the u_k by their asymptotic value $u_k = 1$ for all k. The recursion relation (37) reads now

$$< |a_m|^2 >_\hbar = \frac{g}{m} \sum_{k=1}^{m} < |a_{m-k}|^2 >_\hbar . \tag{38}$$

This is solved by

$$< |a_m|^2 >_\hbar = \begin{pmatrix} m+g-1 \\ g-1 \end{pmatrix} . \tag{39}$$

as can be checked by direct substitution. This solution can be used only for $m < \Lambda/2$. In the range $m > \Lambda/2$ it should be augmented by the identity

$$< |a_l|^2 >_\hbar = < |a_{\Lambda-l}|^2 >_\hbar \tag{40}$$

which is a direct consequence of unitarity. These results can be compared with the results of RMT (11).

Systems *without* time reversal symmetry (TRS) have $g = 1$, and $< |a_m|^2 >_\hbar = 1$. Thus, the semiclassical result coincides with the prediction of the theory for for the CUE, (see (11)), $< |a_m|^2 >_{\beta=2} = 1$.

In chaotic systems *with* TRS, $g = 2$ and

$$< |a_l|^2 >_\hbar \ \approx \ \begin{cases} 1+l & \text{for} \ \ 1 < l \leq \Lambda/2 \\ 1+\Lambda-l & \text{for} \ \ \Lambda > l \geq \Lambda/2 \end{cases} . \tag{41}$$

This expression does not reproduce the RMT result for the COE case ([6])

$$< |a_l|^2 >_{COE} \ = \ 1 + l\frac{\Lambda}{\Lambda+1} - l^2 \frac{1}{\Lambda+1} \tag{42}$$

However, for large Λ, where the semiclassical approximation is justified, the semiclassical result agrees with the exact expression in a domain of l values of size $\sqrt{\Lambda}$ in the vicinity of the end points of the l interval, $l = 0$ and $l = \Lambda$. The deterioration of the quality of the agreement between the semiclassical and the RMT expressions when TRS is imposed is typical, and it is an enigma in the field of quantum chaos [3].

So far, we discussed systems for which all transients die out on a fast time scale, which was imposed by setting $u_k = \mathrm{tr} U^k = 1$ for all k. This is possible only when one eigenvalue

of U is 1 and all the rest vanish. In generic systems, the spectrum is not degenerate in this extreme way. Rather, beside the eigenvalue 1 which corresponds to the conserved phase-space measure, the spectrum is in the interval $[0,1]$, and it accumulates at 0. The rate of decay of transients is determined by the magnitude of the eigenvalues of U which are less then 1. To get the leading correction due to the non vanishing eigenvalues of U, one can expand the recursion relation (37) to first order in $\mu = u_1 - 1$. One obtains in this way recursion relations for the correction to $< |a_m|^2 >_{hbar}$. They are particularly simple for the cases with $g = 1, 2$ and the corrected coefficients are

$$\begin{aligned} < |a_l|^2 >_{hbar} &= 1 + \mu \qquad \text{for} \quad g = 1 \\ &= 1 + l + 2\mu l \quad \text{for} \quad g = 2 \end{aligned} \tag{43}$$

The symmetry $< |a_l|^2 >_{hbar} = < |a_{\Lambda - l}|^2 >_{hbar}$ should be implemented for $l > \Lambda/2$. Recently, the Essen group studied numerically the variances of the coefficients of the characteristic polynomial for the quantum kicked top [6]. They checked systems with and without TRS, and their numerical results show systematic deviations from the RMT predictions which are consistent with the expressions (44). The numerical results for the case without TRS is particularly convincing.

B. Classical Integrable dynamics

For integrable maps we have (24)

$$< |s_n|^2 >_{\hbar} \approx \Lambda \tag{44}$$

The resulting recursion relations for the coefficients $< |a_m|^2 >_{\hbar}$ are

$$< |a_m|^2 >_{\hbar} = \frac{\Lambda}{m} \sum_{k=1}^{m} \frac{< |a_{m-k}|^2 >_{\hbar}}{k} \tag{45}$$

We were not able to find a close form for the solution of this equation. However, to leading order $< |a_m|^2 >_{\hbar} \approx \frac{\Lambda^m}{m!}$ which coincides with the leading term of the result for the Poisson ensemble (12). As can be checked by direct evaluation, the deviation between the semiclassical and the exact values occurs already in the expression for $< |a_{m=2}|^2 >_{\hbar}$.

IV. CONCLUSIONS

In this work we derived the semiclassical theory for the autocorrelation functions of spectral determinants, by obtaining explicit expressions for their Fourier coefficients. The semiclassical theory provided two basic ingredients - it supported the conjecture (♠) that the semiclassical ensemble of s_n form a random Gaussian ensemble, and, it provided the variances $< |s_n|^2 >_\hbar$ for the various cases under study. The comparison with the corresponding RMT results is not uniformly successful, and we would like to correlate the degree of success to the accuracy by which the semiclassical theory provides the two ingredients mentioned above.

For *chaotic* systems *without* TRS, the assumption (♠) is well founded, and the semiclassical expression for $< |s_n|^2 >_\hbar$ coincides with $< |s_n|^2 >_{\beta=2}$. Hence the semiclassical theory for the autocorrelation coefficients matches exactly the RMT result.

For *chaotic* systems *with* TRS, the assumption (♠) is well founded, but the semiclassical expression for $< |s_n|^2 >_\hbar$ agrees with $< |s_n|^2 >_{\beta=1}$ only in the low n domain. Hence the semiclassical and the RMT results agree to leading order only.

For integrable system the assumption (♠) is not so well founded, but the semiclassical expression for $< |s_n|^2 >_\hbar$ agrees with $< |s_n|^2 >_{\beta=0}$, resulting again in agreement only to leading order between the semiclassical and the RMT results.

The last point which should be mentioned is the natural occurrence of the classical Ruelle ζ in the theory. This phenomenon was recently observed by several groups which studied the correspondence between quantum chaos and RMT using other statistical measures [4], [5]. We would like to emphasize, whoever, that it is not correct to use ζ_{Ruelle} directly in the quantum theory, (see (29)). Rather, the quantum character of the problem must be introduced by truncating the coefficients at $l = \Lambda/2$, and imposing unitarity via the relation $< |a_l|^2 >=< |a_{\Lambda-l}|^2 >$. Remembering that Λ plays the rôle of the Heisenberg time in the present theory, we can interpret the above statement as the analogue of the "Riemann Siegel look-alike" [9] symmetry which is a basic ingredient in the semiclassical theory of spectral determinants.

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